Utilizing the Financial Calculator for Graduated Annuities: Reconciling Present Value, Future Value, Compound Rate, and Discount Rate

Su-Jane Chen
Metropolitan State University of Denver
chens@msudenver.edu

A graduated or growing annuity is similar to an annuity except for the fact that its periodical cash flows increase over time at a constant rate. Graduated annuities have many applications, notably for investors whose periodical savings increase at a constant rate to reflect their elevated earnings power over time and retirees whose periodical withdrawals need to rise to keep up with anticipated cost of living. A financial calculator would come handy if we can utilize it to derive the present value, future value, or the initial payment of graduated annuities as we do for annuities.

Graduated annuities, unlike annuities, involve two rates, the growth rate and the interest rate. Both rates have to be incorporated to yield a net rate to enlist the usage of a financial calculator. For the derivation of the precise net rate, I_c, the following formula, in reflection of the impact of compounding and its ensuing multiplicative process, has to be used:

$$\boldsymbol{I_c = \frac{(1+g)}{(1+r)} - 1} \text{ or } \boldsymbol{(1 + I_c) = \frac{(1+g)}{(1+r)}} \tag{1}$$

where I_c is the net compound rate, g is the growth rate, and r is the interest rate.
Alternatively, the net rate can be defined as I_d below to acknowledge the impact of discounting:

$$\boldsymbol{I_d = \frac{(1+r)}{(1+g)} - 1} \text{ or } \boldsymbol{(1 + I_d) = \frac{(1+r)}{(1+g)}} \tag{2}$$

where I_d is the net discount rate.

Graduated annuities have not received much attention in finance literature or textbooks. Hall (1996) presents the application of the financial calculator to two graduated annuity examples without much regard to the rationale behind the mechanism. Bagamery (2011) proposes a transformation of nominal cash flows into real cash flows for the financial calculator to be applicable to graduated annuity problems. For textbooks that cover the topic, they either rely on a complex formula for illustration (Brealey, Myers, Allen, and Edmans, 2023; Ross, Westerfield, Jaffe, and Jordan, 2016) or focus on mechanical derivation of the initial graduated annuity payment without much intuitive explanation (Brigham and Daves, 2019; Brigham and Ehrhardt, 2017). In contrast, we methodically construct a conceptual foundation for the financial calculator application to graduated annuities. Moreover, we introduce the net compound rate as an alternative to the conventional net discount rate as the entry for I in the financial calculator. We further present numerical examples to demonstrate the utilization of the financial calculator in handling graduated annuities and to show how the two rate modification methods can be used interchangeably with ease.

MATHEMATICAL WORK

A graduated annuity due is one where the first cash flow occurs right away. We will proceed with this graduated annuity type.

Let's define $PVGA_{due}$, Pmt_1, g, r, and N, respectively, as the present value of a graduated annuity due, the first cash flow, the growth rate, the interest rate, and the number of cash flows. Equation (3) then formulates the mathematical derivation of $PVGA_{due}$.

$$\boldsymbol{PVGA_{due} = Pmt_1 * [\frac{(1+g)^0}{(1+r)^0} + \frac{(1+g)^1}{(1+r)^1} + \frac{(1+g)^2}{(1+r)^2} + \cdots + \frac{(1+g)^{N-1}}{(1+r)^{N-1}}]} \tag{3}$$

Recalling Equation (1) for $(1+I_c)$, Equation (3) can be expressed as Equation (4).

$$\boldsymbol{PVGA_{due} = Pmt_1 * [(1 + I_c)^0 + (1 + I_c)^1 + (1 + I_c)^2 + \cdots + (1 + I_c)^{N-1}]} \tag{4}$$

As we can see, the expression inside the brackets above captures the value of an ordinary annuity of $1 compounded at an I_c rate for N periods. As a result, Equation (4) can be rewritten as

$$\boldsymbol{PVGA_{due} = FVA_{I_c,N}} \tag{5}$$

where $FVA_{Ic,N}$ represents the future value of an ordinary annuity in the amount of Pmt_1 compounded at a rate of I_c with N as the number of payments.

We can now derive the present value of a graduated annuity with a financial calculator by getting the future value of an ordinary annuity with the net compound rate, I_c, as I, the initial payment, Pmt_1, as PMT, and the number of payments, N, as N. For future reference, we refer to this as the net compound rate approach.

Alternatively, we can rewrite Equation (3) as the following:

$$PVGA_{due} = Pmt_1 * \left[\frac{1}{\left[\frac{(1+r)^0}{(1+g)^0}\right]} + \frac{1}{\left[\frac{(1+r)^1}{(1+g)^1}\right]} + \frac{1}{\left[\frac{(1+r)^2}{(1+g)^2}\right]} + \cdots + \frac{1}{\left[\frac{(1+r)^{N-1}}{(1+g)^{N-1}}\right]} \right] \quad (6)$$

Recalling Equation (2) for (1+I_d), we can reset $PVGA_{due}$ in Equation (6) as below.

$$PVGA_{due} = Pmt_1 * \left[\frac{1}{(1+I_d)^0} + \frac{1}{(1+I_d)^1} + \frac{1}{(1+I_d)^2} + \cdots + \frac{1}{(1+I_d)^{N-1}} \right] \quad (7)$$

The expression inside the brackets above reflects the value of an annuity due of \$1 discounted at an I_d rate for N periods. Subsequently, Equation (8) emerges.

$$PVGA_{due} = PVA_{due,I_d,N} \quad (8)$$

where $PVA_{due,Id,N}$ represents the present value of an annuity due in the amount of Pmt_1 discounted at a rate of I_d with a number of payments of N.

We can now obtain the present value of a graduated annuity due with a financial calculator by solving for the present value of an annuity due with the net discount rate, I_d, as I, the initial payment, Pmt_1, as PMT, and the number of payments, N, as N. For discussion below, we label this as the net discount rate approach.

PRESENT VALUE OF A GRADUATED ANNUITY DUE

Next, we adopt a numerical example to demonstrate the implementation of the two presented approaches for the derivation of the present value of a graduated annuity due with a financial calculator. We assume an initial payment of \$1,000, three payments, and 8% and 35%, respectively, for the growth rate and the interest rate. This will result in a net compound rate of −20% and a net discount rate of 25%.

Based on the net compound rate approach summarized in Equation (5), we can derive the present value of a graduated annuity, \$2,440, by entering 3 as N, the net compound rate of −20% for I, 0 as PV, and −\$1,000 for PMT, and solving for FV. According to the net discount rate approach reflected in Equation (8), we can obtain the same amount, \$2,440, by entering the net discount rate, 25%, as I, resetting FV to 0, switching PMT mode to BEGIN, and solving for PV.

INITIAL PAYMENT

Another occasion where the financial calculator application to a graduated annuity due comes handy is when a lottery winner chooses the "lottery annuity" option. Under this payout option, the total is to be distributed over 29 years with 30 annual payments, where the first payment is to be received today and each payment afterwards is set at 5% larger than the last one, a perfect example of a graduated annuity due with g and r equal to 5% and 0%, respectively. Following the two illustrated approaches, we modify the two rates to derive the net compound rate of 5% and the net discount rate of approximately −4.7619%. Assuming a winning prize of \$100 million, the employment of the financial calculator under the two approaches results in \$1,505,143.51 as the initial payment of the graduated annuity due.

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